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Symmetries of decimation invariant sequences and digit sets

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Abstract

A bi-infinite sequence is called p-decimation invariant if all p-decimations of it reproduce the sequence albeit with a shift. In this paper we discuss symmetry properties of decimation invariant sequences. A symmetry is a composition of a translation and a reflection. We establish the existence of translation invariant, i.e., periodic, decimation invariant sequences. Moreover, we prove that there exist decimation invariant sequences which are left-periodic and right-periodic, i.e., they are partially translation invariant. We present several criteria for the existence of decimation invariant sequences with additional periodicity properties. Finally, we discuss the existence of decimation invariant sequences that are invariant under reflections. Moreover, in passing we demonstrate that properties of decimation invariant sequences are linked with properties of certain digit sets. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

Coarse-graining invariant sequences generated by cellular automata were introduced in [5] and further investigated in [3, 4]. A special case of coarse-graining invariant sequences (without referring to a cellular automaton) are the decimation invariant sequences as introduced in [7]. Loosely speaking, one can say that a decimation invariant sequence reproduces itself after deleting some of their elements.

In this paper we study one-dimensional bidirectionally infinite sequences with values in $\mathbb{Z}$, which is no restriction. As a matter of fact, as it will turn out, it is more

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convenient to do so. We consider p-decimation invariant sequences which have an additional symmetry. The possible symmetries are given by reflections or translations. If a sequence is invariant under translations, then the sequence is periodic.

The paper is organized as follows. After having introduced some basic facts in Section 2, we begin with the definition of decimation invariant sequences. Although the definition given here seems to be slightly more general than the one given in [7], we prove that both definitions are equivalent. Furthermore, we show how decimation invariant sequences are related to Mahler equations. In fact, we prove that every decimation invariant sequence is a solution of a special kind of Mahler equation, a so-called *decimation invariance equation*. Furthermore, every solution of a decimation invariance equation is a decimation invariant sequence.

Section 4 is devoted to a closer study of decimation invariant sequences which are periodic, a subject which was touched upon in [7]. We give a new and improved proof of Theorem 3 in [7] concerning the necessary condition for the periodicity of a decimation invariant sequence. We discuss the possible periods of a decimation invariant sequence and state some sufficient conditions for periodicity.

In Section 5, we try to find decimation invariance equations such that every solution of the decimation invariance equation is periodic. We show that there is a link between complete digit sets, [2, 14, 16, 17], and with a weaker concept which we introduce as quasi-complete digit sets. Quasi-complete digit sets correspond to decimation invariance equation having only periodic solutions. Several properties of quasi-complete digit sets are discussed.

In Section 6, we discuss the existence and non-existence of decimation invariant sequences which are periodic to the left or right, i.e., the sequences have a partial symmetry.

Finally, in Section 7 we study decimation invariant sequences which are symmetric under reflections. We discuss in detail a conjecture stated in [6] and show in which case the conjecture is true or not true.

2. Basic facts

We will be dealing with bidirectionally infinite sequences $(f_j)_{j\in\mathbb{Z}}$ with values in $\mathbb{Z}$. It will be convenient to represent such a sequence as a formal Laurent-series

$$f(X) = \sum_{j\in\mathbb{Z}} f_j X^j$$

and we will in that sense speak of the sequence $f(X)$. The set of all sequences will then also be identified with the set $\mathbb{Z}(X)$ of all Laurent-series. Note that we can add two series and multiply a series with a ring element. Note further that we cannot multiply two series $f(X)$, $g(X)$ unless there exists a $k\in\mathbb{Z}$ such that $f_l=0$ or $g_l=0$ for all $l<k$. A Laurent-series having this property is simply called a power series, and the corresponding sequence has an infinite number of zeroes to its left side.

- For $\alpha \in \mathbb{Z}$ the shifted sequence, denoted as $X^\alpha f(X)$ is defined as

$$X^\alpha f(X) = \sum_{j\in\mathbb{Z}} f_j X^{j+\alpha}.$$

  If $\alpha > 0$, then $X^\alpha f(X)$ is obtained by shifting $f(X)$ over α-units to the right. For $\alpha < 0$ the sequence is shifted α units to the left.
- If p is an integer different from 0 and if $f(X) = \sum_{j\in\mathbb{Z}} f_j X^j$ is a sequence, then the p-inflation, $f(X^p)$, of $f(X)$ denotes the sequence

$$f(X^p) = \sum_{j\in\mathbb{Z}} f_j X^{pj}.$$

  The p-inflation of a sequence inserts $|p| - 1$ zeroes between each element of the sequence. If $p < 0$, then the inflated sequence is reflected about zero.

Let p be an integer such that $|p| \geqslant 2$ and $l \in \mathbb{Z}$. The map $\partial_{l,p} : \mathbb{Z}(X) \to \mathbb{Z}(X)$ defined as

$$\partial_{l,p}(f)(X) = \partial_{l,p}(f(X)) = \partial_{l,p}\left(\sum_{j\in\mathbb{Z}} f_j X^j\right) = \sum_{j\in\mathbb{Z}} f_{pj+l} X^j \tag{1}$$

is called l-decimation with respect to p. We agree to write ∂_l for an l-decimation $\partial_{l,p}$ if there is no risk of confusion. The following basic properties will be useful.

- If $f(X)$ and $g(X)$ are sequences, then

$$\partial_w(f(X) + g(X)) = \partial_w(f(X)) + \partial_w(g(X)) \tag{2}$$

- If $f(X) = X^{pl+j} f(X^p)$, where $l, j \in \mathbb{Z}$, then for all $i \in \mathbb{Z}$

$$\partial_{i,p}(f)(X) = \begin{cases} X^{l-l'} f(X) & \text{if } i = pl' + j, \\ 0 & \text{otherwise} \end{cases}$$

  holds.

Let $p \in \mathbb{Z}$ such that $|p| \geqslant 2$. A finite set $W = \{w_0, \ldots, w_{|p|-1}\}$ is called a p-residue set if for any $j \in \mathbb{Z}$ there exist unique $j' \in \mathbb{Z}$ and $w \in W$ such that

$$j = pj' + w$$

holds.

Decimations play a central rôle in the theory of automatic sequences (see [1] for more details). In the context of automatic sequences one considers usually the decimations $\partial_{j,p}$, where $p \geqslant 2$ and $j \in \{0, \ldots, p-1\}$ restricted to sequences $(a_j)_{j\in\mathbb{N}}$. In our context we need a more flexible definition of automaticity which was developed in [12].

Let $|p| \geqslant 2$ be a natural number and let $W = \{w_0, \ldots, w_{|p|-1}\}$ be a p-residue set. For $f(X) \in \mathbb{Z}(X)$ the set

$$\ker_W(f(X)) = \{\partial_{w_{i_1}} \circ \cdots \circ \partial_{w_{i_n}}(f(X)) \mid w_{i_1}, \ldots, w_{i_n} \in W,\ n \in \mathbb{N}\}$$

is called the W-*kernel* of $f(X)$. For $n = 0$ no decimations are applied, i.e., $f(X) \in \ker_W(f(X))$. The sequence $f(X)$ is called W-automatic if $\ker_W(f(X))$ is a finite set.

Due to Theorem 3.2 in [12] the finiteness of the W-kernel does not depend on the p-residue set W. It is therefore justified to speak of the p-automaticity of a sequence.

The following observation is trivial but important:

Let $p \in \mathbb{Z}$ such that $|p| \geqslant 2$ and let W be a p-residue set, then for all $f(X)$

$$f(X) = \sum_{w \in W} X^w \partial_w(f)(X^p).$$

Note the difference between $\partial_w(f)(X^p)$ and $\partial_w(f(X^p))$. The former computes the decimation and then applies the p-inflation to the decimated sequence. The latter applies the p-inflation to $f(X)$ and then computes the decimation.

For the study of decimation invariant sequences it is important to study the dynamics of the map Ξ which will be defined now, see also its importance in [12].

Definition 1. Let W be a p-residue set. The maps $\Xi : \mathbb{Z} \to \mathbb{Z}$ and $\zeta : \mathbb{Z} \to W$ defined as

$$j = p\Xi(j) + \zeta(j)$$

are called *image-part map* and *remainder-part map*, respectively.

As usual, Ξ^n denotes the nth iterate of Ξ. The set of fixed points of Ξ is denoted by $\mathrm{Fix}(\Xi)$ and the set of periodic points is denoted by $\mathrm{Per}(\Xi)$. The point $j \in \mathbb{Z}$ is called preperiodic (w.r.t. Ξ) if there exists an N such that $\Xi^N(j) \in \mathrm{Per}(\Xi)$.

Lemma 1. *If W is a p-residue set, where $|p| \geqslant 2$, and Ξ is its associated image-part map, then every $j \in \mathbb{Z}$ is preperiodic (w.r.t. Ξ).*

Proof. Since $l = p\Xi(l) + \zeta(l)$ it follows that

$$|\Xi(l)| \leqslant \frac{|l| + |\zeta(l)|}{|p|}$$

for all $l \in \mathbb{Z}$. Therefore, for $c = \max\{|w| \mid w \in W\}$ we have

$$|\Xi(l)| \leqslant \frac{|l| + c}{|p|},$$

implying the existence of the positive integer $c' = \lfloor c/(|p|-1|) \rfloor$ such that $\Xi(\mathbb{Z} \cap [-c', c']) \subset [-c', c']$. Moreover, for $l \in \mathbb{Z}$ there exists $M = M(l)$ such that $\Xi^M(l) \in [-c', c']$. Finally, due to the finiteness of $\mathbb{Z} \cap [-c', c']$, there exists an $N = N(l)$ such that $\Xi^N(l)$ is a periodic point. $\square$

Example 1. Let $p = 5$ and consider the 5-residue set $W = \{0, 8, 16, 24, 27\}$. Following the above proof we see that $c' = \lfloor \frac{27}{4} \rfloor = 6$. Thus, all periodic points of the associated image-part map Ξ are contained in the interval $[-6, 6]$. In fact, a closer look reveals that $\mathrm{Fix}(\Xi) = \{-6, -4, -2, 0\}$ and there is a period two orbit given by $\Xi(-5) = -1$ and $\Xi(-1) = -5$. Thus $\mathrm{Per}(\Xi) = \{-6, -5, -4, -3, -2, -1, 0\}$.

Motivated by the above example we introduce the notion of a core interval.

Definition 2. Let W be a p-residue set. The *core interval* of W is the smallest interval $C(W) \subset \mathbb{R}$ that contains all periodic points of the image-part map Ξ of W.

Since the periodic points of Ξ are elements of $\mathbb{Z}$, it follows that the endpoints of the core interval are in $\mathbb{Z}$, too.

Example 1 (*continued*). For the above example we have that $C(W) = [-6, 0]$.

The corollary below gives a hint on the size of the core interval. For the proof we need some preparations. Note that the core interval does not change if we consider any iterate of Ξ.

The second iterate Ξ^2 of Ξ can also be considered as the image-part map for the p^2-residue set $W^{(2)} = W + pW = \{v + pw \mid v, w \in W\}$. In a similar way we can define $W^{(n)} = W + pW + \cdots + p^{n-1}W$ and we have that Ξ^n is the image-part map w.r.t. $W^{(n)}$.

Corollary 1. *Let W be a p-residue set. If $W^{(2)} = \{w_0 < \cdots < w_{p^2-1}\}$ denotes the associated p^2-residue set, then*

$$C(W) \subseteq \left[\left\lceil -\frac{w_{p^2-1}}{p^2-1} \right\rceil, \left\lfloor -\frac{w_0}{p^2-1} \right\rfloor\right].$$

Proof. Let Ξ, ζ be the image-part and remainder map w.r.t. $W^{(2)}$. For any $j \in \mathbb{Z}$ we have $j = p^2 \Xi(j) + \zeta(j)$ and therefore

$$\Xi(j) = \frac{j - \zeta(j)}{p^2},$$

which gives

$$L(j) = \frac{j - w_{p^2-1}}{p^2} \leqslant \Xi(j) \leqslant \frac{j - w_0}{p^2} = R(j)$$

for all $j \in \mathbb{Z}$. Iterating n times gives

$$L^n(j) \leqslant \Xi^n(j) \leqslant R^n(j)$$

for all $n \in \mathbb{N}$ and $j \in \mathbb{Z}$. Since $\lim_{n\to\infty} L^n(j) = -w_{p^2-1}/(p^2-1)$ and $\lim_{n\to\infty} R^n(j) = -w_0/(p^2-1)$, we conclude that there exist an $N = N(j)$ such that

$$\Xi^n(j) \in \left[-\frac{w_{p^2-1}}{p^2-1}, -\frac{w_0}{p^2-1}\right]$$

for all $n \geqslant N$. As $j \in \mathrm{Per}(\Xi)$ implies that $\Xi^n(j) \in \mathrm{Per}(\Xi)$ for all n, it also follows that

$$\mathrm{Per}(\Xi) \subseteq \left[-\frac{w_{p^2-1}}{p^2-1}, -\frac{w_0}{p^2-1}\right].$$

Using the fact that Per $(\Xi) \subset \mathbb{Z}$, we arrive at the desired result. $\square$

If p is a positive number and if $W=\{w_0<\cdots<w_{p-1}\}$ is a p-residue set, then one has

$$C(W)\subseteq\left[\left\lceil-\frac{w_{p-1}}{p-1}\right\rceil,\left\lfloor-\frac{w_0}{p-1}\right\rfloor\right].$$

The following observations are trivial but important:

- Let $p\geqslant 2$ and let $W=\{w_0<\cdots<w_{p-1}\}$ be a p-residue set. $w_0/(p-1)\in\mathbb{Z}$ if and only if $-w_0/(p-1)$ is a fixed point of Ξ. Also, $w_{p-1}/(p-1)\in\mathbb{Z}$ if and only if $-w_{p-1}/(p-1)$ is a fixed point of Ξ.
- Let $p\leqslant -2$ and let W be a p-residue set and $W^{(2)}=\{w_0<\cdots<w_{p^2-1}\}$. The point $-w_0/(p^2-1)$ is a fixed point of Ξ^2 if and only if $-w_0/(p^2-1)\in\mathbb{Z}$. The same is true for $w_{p^2-1}/(p^2-1)$.

Example 2. Let $W=\{0,8,16,24,37\}$, then Ξ has the fixed points $\{0,-2,-4,-6\}$ and a period two cycle $\{-1,-5\}$, thus $C(W)=[-6,0]$. The above method yields

$$C(W)\subset[-9,0].$$

As already noticed, any $l\in\mathbb{Z}$ is preperiodic w.r.t. Ξ. We say that l_1 and $l_2\in\mathbb{Z}$ are Ξ-equivalent, denoted as $l_1\sim_\Xi l_2$, if there exist $N_1,N_2\in\mathbb{N}$ such that

$$\Xi^{N_1}(l_1)=\Xi^{N_2}(l_2).$$

It is obvious that this does indeed define an equivalence relation. Two points are equivalent if they belong eventually to the same periodic orbit of Ξ. We denote the quotient $\mathbb{Z}/\sim_\Xi$ by $P(\Xi)$.

Example 1 (*continued*). For $p=5$ and $W=\{0,8,16,24,27\}$ we have that $|P(\Xi)|=5$, since Ξ has four fixed points and one periodic point of period two.

If $s\in\mathbb{Z}$, then we set $\Xi^{-1}(s)=\{ps+w\mid w\in W\}$.

Definition 3. Let $p\in\mathbb{Z}$ such that $|p|\geqslant 2$ and let W be a p-residue set. A subset $S\subset\mathbb{Z}$ is called *generating* (w.r.t. Ξ) if

(1) $\bigcup_{n\in\mathbb{N}}\Xi^{-n}(S)=\mathbb{Z}$,

(2) If $S'\subset S$ such that S' satisfies (1), then $S=S'$.

The first condition guarantees that the set S generates $\mathbb{Z}$, while the second condition establishes a minimality property of a generating set.

Example 1 (*continued*). For $p=5$ and $W=\{0,8,16,24,27\}$ the set of periodic points of Ξ is $\mathrm{Per}(\Xi)=\{-6,-5,-4,-2,-1,0\}$, where $\mathrm{Fix}(\Xi)=\{-6,-4,-2,0\}$, and -1 together with -5 form a period two orbit. Thus, we have the two generating sets

$$S_1=\{-6,-4,-2,0\}\cup\{-1\},$$
$$S_2=\{-6,-4,-2,0\}\cup\{-5\}.$$

Since every $l \in \mathbb{Z}$ is preperiodic w.r.t. Ξ, it follows that a generating set must be a subset of the periodic points. Due to the minimal property, a generating set contains as many points as there exist different periodic orbits. Thus $|S| = |P(\Xi)|$.

3. Decimation invariant sequences

Decimation invariance sequences (DI-sequences), introduced in [7], are sequences which after a decimation remain the same sequence (modulo a shift). After giving a formal definition, we will show that a DI-sequence satisfies a special kind of Mahler equation (called DI-equation), implying that DI-sequences are automatic.

Mahler equations provide an algebraic tool to describe automatic sequences complementary to the rather combinatorial description presented above. Such algebraic description appeared in [9, 10] where it was shown that a p-automatic sequence with values in a finite field of characteristic p, p a prime number, satisfies a Mahler equation.

As it will turn out, the study of the set of solutions of a DI-equation is closely related to the properties of a image-part map Ξ (w.r.t. a p-residue set W).

Definition 4. Let $p \in \mathbb{Z}$ such that $|p| \geqslant 2$ and let $W = \{w_0, \dots, w_{|p|-1}\}$ be a p-residue set. A sequence $f(X)$ is called (p, W)-*decimation invariant* if for each $w \in W$ there exists a $\kappa_w \in \mathbb{Z}$ such that

$$\partial_w(f(X)) = X^{\kappa_w} f(X)$$

holds.

Lemma 2. *If $f(X)$ is (p, W)-invariant and V is a p-residue set, then $f(X)$ is also (p, V)-invariant.*

Proof. This follows from

$$f(X) = \sum_{w \in W} X^w \partial_w(f)(X^p)$$

and the (p, W)-decimation invariance of $f(X)$ implies

$$f(X) = \sum_{w \in W} X^w X^{p\kappa_w} f(X^p) = \sum_{w \in W} X^{w+p\kappa_w} f(X^p).$$

The remainder- and image-part maps (w.r.t. V) are denoted as ζ and Ξ, respectively. In terms of ζ and Ξ, we get

$$f(X) = \sum_{w \in W} X^{\zeta(w+p\kappa_w)} X^{p\Xi(w+p\kappa_w)} f(X^p).$$

Since for each $v \in V$ there exists a unique $w \in W$ such that $v = \zeta(w + p\kappa_w)$, we have

$$\partial_v(f(X)) = X^{\Xi(w+p\kappa_w)} f(X).$$

Thus the sequence is (p, V)-decimation invariant. $\square$

From now on we say that a sequence is *p-decimation invariant*, or it is a p-DI-sequence, or simply a DI-sequence.

In [7] a sequence was called $(p,\underline{\kappa})$-decimation invariant if there exists a $\underline{\kappa}=(\kappa_0,\cdots,\kappa_{|p|-1})\in\mathbb{Z}^{|p|}$ such that $f(X)$ satisfies a *decimation invariance equation*, namely

$$\partial_i(f)(X) = X^{-\kappa_i} f(X) \tag{3}$$

holds for all $i\in\{0,\dots,|p|-1\}$, which is equivalent to the fact that

$$(f_{jp+i})_{j\in\mathbb{Z}} = (f_{j+\kappa_i})_{j\in\mathbb{Z}} \tag{4}$$

holds for all $i\in\{0,\dots,|p|-1\}$. I.e., any i-decimation of a sequence reproduces the original sequence, albeit with a possible shift.

Due to our above observations, the definition in [7] is equivalent to Definition 4.

Theorem 1. *The sequence $f(X)\in\mathbb{Z}(X)$ is p-decimation invariant if and only if there exist a p-residue set W such that $f(X)$ satisfies the Mahler equation*

$$f(X) = \left(\sum_{w\in W} X^w\right) f(X^p). \tag{5}$$

Proof. Suppose $f(X)$ satisfies (5). Then it follows for all $v\in W$

$$\partial_v(f(X)) = \partial_v\left(\left(\sum_{w\in W} X^w\right) f(X^p)\right) = f(X)$$

holds. In other words, $f(X)$ is (p,W)-decimation invariant.

If $f(X)$ is a p-DI sequence invariant, then there exist a p-residue set V such that

$$\partial_v(f)(X) = X^{\kappa_v} f(X)$$

holds for all $v\in V$. We conclude that

$$f(X) = \sum_{v\in V} X^v \partial_v(f)(X^p) = \left(\sum_{v\in V} X^{v+p\kappa_V}\right) f(X^p),$$

which shows that $f(X)$ satisfies an equation of the desired type for the p-residue set $W=\{v+p\kappa_v \mid v\in V\}$. $\square$

In other words, any solution of a Mahler equation of the above type gives a p-decimation invariant sequence, and vice versa.

Note that by the above theorem, for a p-DI sequence $f(X)$ there exists a particular p-residue set W such that

$$\partial_w(f(X)) = f(X)$$

for all $w\in W$. If we want to emphasize that we use this particular p-residue set, we say that $f(X)$ is a DI-sequence w.r.t. W.

By Theorem 6.3 in [12] (see also [8, 10, 11, 18, 19] for related results), it follows that DI-sequences are p-automatic. In [7], algorithms for computing the kernel can be found.

Thus, a study of p-decimation invariant sequences is equivalent to a study of the solutions of Mahler equations of the type

$$f(X) = (X^{w_0} + \cdots + X^{w_{|p|-1}})f(X^p), \tag{6}$$

where the set $W = \{w_0, \ldots, w_{|p|-1}\}$ is a p-residue set.

An equation of the above type is called *DI-equation* (*w.r.t.* p) (decimation invariance equation). It is equivalent to

$$(f_{jp+w})_{j\in\mathbb{Z}} = (f_j)_{j\in\mathbb{Z}} \tag{7}$$

for all $w \in W$.

If W is a given p-residue set, then

$$Q_W(X) = \sum_{w\in W} X^w$$

denotes the polynomial of the corresponding DI-equation.

The next lemma shows the invariance of DI-sequences under certain operations.

Lemma 3. *If $f(X)$ is a p-DI-sequence, then the shifted sequence $X^\alpha f(X)$, $\alpha \in \mathbb{Z}$ and the inverted sequence $f(X^{-1})$ are DI-sequences.*

Proof. If $f(X)$ is a DI-sequence, then there exist a p-residue set W such that $f(X) = Q_W(X)f(X^p)$. Thus, we have

$$X^\alpha f(X) = X^\alpha Q_W(X) X^{-p\alpha} X^{p\alpha} f(X^p)$$

which yields that $g(X) = X^\alpha f(X)$ satisfies

$$g(X) = Q_{W+(1-p)\alpha}(X) g(X^p),$$

i.e, $g(X)$ is a DI-sequence w.r.t. $W + (1-p)\alpha$.

The second assertion follows similarly. We have $f(X) = Q_W(X)f(X^p)$ and thus $f(X^{-1}) = Q(X^{-1})f(X^{-p})$ which shows that $f(X^{-1})$ is a DI-sequence w.r.t. $-W$. □

The meaning of the above lemma is clear. If we translate the residue set by $\alpha(1-p)$, then the solutions of the corresponding DI-equation will be shifted by α.

In [7], the authors show how to construct all solutions of the above DI- equations using the concept of a *seed*, i.e., a solution of the DI-equations restricted to the core interval, from which it is possible to construct the whole solution in a recursive way. We now relate this to the notion of a generating set (w.r.t. Ξ), see Definition 3.

Theorem 2. *Let $p \in \mathbb{Z}$ such that $|p| \geqslant 2$ and let $W \subset \mathbb{Z}$ be a p-residue set. Then the solutions of the DI-equation*

$$f(X) = Q_W(X) f(X^p)$$

are completely determined by their values on a generating set S (w.r.t. image-part map Ξ). i.e., for any $j \in \mathbb{Z}$ there exist $s \in S$ such that $j \in \cup_{n \in \mathbb{N}} \Xi^{-n}(s)$ and f_j is equal to f_s.

Proof. The sequence $f(X) = \sum f_j X^j$ is a solution of the DI-equation if and only if $f_j = f_{pj+w}$ (see (7)) holds for all $j \in \mathbb{Z}$ and all $w \in W$, or, equivalently $f_j = f_{\Xi(j)}$ for all $j \in \mathbb{Z}$. Therefore $f_{j_1} = f_{j_2}$ whenever $j_1 \sim_{\Xi} j_2$. Thus, a solution is completely determined by its values on a generating set. $\square$

As the generating set is a subset of $\mathrm{Per}(\Xi) \subset C(W)$, a generating set corresponds to those elements in the seed (in the sense given in [7]) which can take a value in $\mathbb{Z}$ in an independent way.

Like in [7], a solution $f(X)$ of a DI-equation is called a solution of *maximal diversity* if and only if $|\{f_j \mid j \in \mathbb{Z}\}| = |P(\Xi)|$, i.e., different $\mathbb{Z}$-values are attributed to different elements of a generating set. All other solutions of a DI-equation are obtained from the solution of maximal diversity by assigning the same value to different elements of a generating set.

Example. (1) Let $p = 5$ and $W = \{0, 8, 16, 24, 27\}$. As we have seen $S = \{-6, -4, -2, -1, 0\}$ is a generating set. Thus a solution of the DI-equation

$$f(X) = (1 + X^8 + X^{16} + X^{24} + X^{27}) f(X^5)$$

is completely determined by its values f_s, where $s \in S$: For $j \in \bigcup_{n \in \mathbb{N}} \Xi^{-n}(s)$ we have $f_j = f_s$. The solution of maximal diversity has 5 different values.

(2) Let $p = 5$ and $W = \{-24, -7, 0, 7, 24\}$. The core interval is given by $[-6, 6]$. The periodic points are given by

$$\begin{array}{rlll} -6 & \mapsto -6 & & \\ -5 & \mapsto -1 & \mapsto -5 & \\ -4 & \mapsto 4 & \mapsto -4 & \\ 0 & \mapsto 0 & & \\ 1 & \mapsto 5 & \mapsto 1 & \\ 6 & \mapsto 6 & & \end{array}$$

A generating set S contains $\{-6, 0, 6\}$ and a complete set of representatives of the sets $\{-5, -1\}$, $\{-4, 4\}$, $\{1, 5\}$. Thus a solution of maximal diversity has 6 different values.

(3) Let $W = \{1, 4, 10, 19\}$ be a 4-residue set. The core interval is $C(W) = [-6, -1]$ and Ξ has one periodic orbit of period 6, i.e., $\mathrm{Per}(\Xi) = \{-6, -5, -4, -3, -2, -1\}$. Thus, a generating set is given by any $s \in \mathrm{Per}(\Xi)$. The solution of maximal diversity is the constant sequence.

4. Periodic solutions

In this section, we discuss the existence of periodic DI-sequences. It generalizes and extends a result obtained in [7].

A sequence $f(X)$ is periodic of period d if $f_j = f_{j+d}$ holds for all $j \in \mathbb{Z}$, and the sequence $f(X)$ has minimal period D if $f_j = f_{j+D}$ holds for all $j \in \mathbb{Z}$ and $D \geqslant 1$ is the smallest possible value of all possible values of d.

We begin with a necessary criterion for a DI-equation to have a periodic solution. Then we consider the periodic solutions in more detail and answer the question under which conditions we can establish the existence of a solution of minimal period D.

Theorem 3. *Let $W = \{w_0, \dots, w_{|p|-1}\}$ be a p-residue set and let $f(X) = Q_W(X) f(X^p)$ be the associated DI-equation. If $f(X)$ is a periodic solution of minimal period D, then D is a divisor of* $\gcd(w_0 - w_1, \dots, w_0 - w_{|p|-1}) = \gcd(w_0 - W)$.

Proof. As a first step, we show that $\gcd(D, p) = 1$. Let us suppose that $\gcd(D, p) = d > 1$. Then the period of $\partial_w(f(X))$ is equal to D/d. Since $f(X)$ is a DI-sequence, we have that $\partial_w(f(X)) = f(X)$ is a periodic sequence of minimal period D. That gives a contradiction, and therefore $\gcd(D, p) = 1$.

Since $f(X)$ is of minimal period D, the sequence is already determined by its values f_i for $i = 0, \dots, D-1$. i.e., we can think of $f(X)$ as a map $g : \mathbb{Z}_D \to \mathbb{Z}$, where $\mathbb{Z}_D$ denotes the ring $\mathbb{Z}_D = \mathbb{Z}/D\mathbb{Z} = \{0, \dots, D-1\}$. On the other hand, the DI-invariance implies that $f_j = f_{pj+w}$ for all $j \in \mathbb{Z}$ and all $w \in W$, which in connection with the periodicity leads to $f_j = f_i$, whenever there exist $w \in W$ such that $i \equiv pj + w \bmod D$. In other words, we have to consider the maps $A_w : s \mapsto ps + w$ as maps from $\mathbb{Z}_D$ to $\mathbb{Z}_D$. The map $g : \mathbb{Z}_D \to \mathbb{Z}$ satisfies $g(A_w(s)) = g(s)$ for all $w \in W$. Since $\gcd(p, D) = 1$, each A_w is a bijective map on $\mathbb{Z}_D$. Thus, for the group $G(W) = \langle \{A_w \mid w \in W\} \rangle$ generated by the A_w's, we have

$$g(\gamma(s)) = g(s)$$

for all $\gamma \in G(W)$ and all $s \in \mathbb{Z}_D$.

Since $f(X)$ is of minimal period D, there exists no $t \in \mathbb{Z}_D$, $t \neq 0$, such that

$$g(s + t) = g(s)$$

holds for all s. Now, suppose there exist v, $w \in W$ such that $w \not\equiv v \bmod D$. Let δ be the order of $A_w \in G(W)$, i.e., $A_w^{\delta}(s) = s$ for all $s \in \mathbb{Z}_D$. That implies

$$A_w^{\delta-1}(A_v(s)) = s + p^{\delta-1} v + (1 + \cdots + p^{\delta-2}) w$$

and $\tau = p^{\delta-1} v + (1 + \cdots + p^{\delta-2}) w \neq 0$. Therefore $G(W)$ contains an element of the form $s \mapsto s + \tau$ which implies that the period of $f(X)$ is $\gcd(\tau, D) < D$. This is a contradiction.

Therefore, we conclude that for all pairs $w, v \in W$ we have $v \equiv w \bmod D$, which proves the assertion. $\square$

Theorem 3 corresponds to Theorem 3 in [7], where the criterion for the existence of a periodic solution of the DI-equation was expressed in terms of the vector $\underline{\kappa} = (\kappa_0, \ldots, \kappa_{|p|-1})$. The above proof is of a different (and improved) nature.

Corollary 2. *If W is a p-residue set and D is a divisor of* $\gcd(w_0 - W)$, *then the image-part map Ξ of W satisfies*

$$p\Xi(j) + w_0 \equiv j \bmod D$$

holds for all $j \in \mathbb{Z}$.

Proof. We have to show that $\Xi(j) \equiv \Xi(j + D) \bmod D$ holds for all $j \in \mathbb{Z}$. Note that $\zeta(j) \equiv w_0 \bmod D$ holds for all $j \in \mathbb{Z}$. Since $j = p\Xi(j) + \zeta(j)$, we have that $j + D = p\Xi(j) + D + \zeta(j)$. On the other hand, $j + D = p\Xi(j + D) + \zeta(j + D)$, implying that $p(\Xi(j) - \Xi(j + D)) \equiv 0 \bmod D$. As $\gcd(p, D) = 1$, it follows that $\Xi(j) \equiv \Xi(j + D) \bmod D$ for all $j \in \mathbb{Z}$. The assertion now follows from $j = p\Xi(j) + \zeta(j) \equiv p\Xi(j) + w_0 \bmod D$. $\square$

Definition 5. Let $W = \{w_0, \ldots, w_{p-1}\}$ be a p-residue set such that $q = \gcd(w_0 - W)$. The map $\Lambda : \mathbb{Z}_q \to \mathbb{Z}_q$ defined as $\Lambda(s) = ps + w_0$ is called *pseudo-inverse* of the image-part map Ξ of W.

Combining Theorem 3 and Corollary 2 we can conclude the following. If $f(X)$ is a DI-sequence w.r.t. W and if $f(X)$ is of minimal period D, then, by Theorem 3, D is a divisor of $\gcd(w_0 - W)$ and by Corollary 2 we have that

$$\Lambda(\Xi(j)) \equiv j \bmod D$$

holds for all $j \in \mathbb{Z}$ (the reason for calling Λ the pseudo-inverse of Ξ). In other words, a periodic DI-sequence is completely determined by the map $\Lambda(s) = ps + w_0$ on $\mathbb{Z}_D$.

Therefore, the study of periodic solutions of a DI-equation is reduced to a study of the orbit structure of the pseudo-inverse of Ξ.

Lemma 4. *If Ξ is the image-part map for the p-residue set W, then*

$$|\mathrm{Per}(\Xi)| \geqslant q,$$

where $q = \gcd(w_0 - W)$.

Proof. We consider the pseudo-inverse $\Lambda(s) = ps + w_0$. Corollary 2 yields $\Lambda^n(\Xi^n(j)) \equiv j \bmod q$ for all $n \in \mathbb{N}$ and all $j \in \mathbb{Z}$. Since $\Lambda : \mathbb{Z}_q \to \mathbb{Z}_q$ is invertible, there exist an $N \geqslant 1$ such that $\Lambda^N(s)$ is the identity map on $\mathbb{Z}_q$. This gives $\Lambda^N(\Xi^N(j)) \equiv j \bmod q$ which is the same as $\Xi^N(j) \equiv j \bmod q$. Thus Ξ^N preserves the residue class $(\bmod q)$ of j. Since any point $j \in \mathbb{Z}$ is eventually periodic (w.r.t. Ξ), any residue class represents a periodic point of Ξ. $\square$

Like for the map Ξ, we introduce the quotient $P(\Lambda) = \mathbb{Z}_q/{\sim}\Lambda$ as the set of equivalence classes given by the equivalence relation $s_1 \sim_\Lambda s_2$ if there exist $N_1, N_2 \in \mathbb{N}$ such that $\Lambda^{N_1}(s_1) = \Lambda^{N_2}(s_2)$. Since $\Lambda : \mathbb{Z}_q \to \mathbb{Z}_q$ is bijective, the quotient $P(\Lambda)$ can be considered as the set of different periodic orbits of Λ, where any $s \in \mathbb{Z}_q$ belongs to exactly one periodic orbit.

Theorem 4. *Let W be a p-residue set and let $q = \gcd(w_0 - W)$. The solutions of the DI-equation*

$$f(X) = Q_W(X) f(X^p)$$

are all periodic with period q if and only if $|P(\Xi)| = |P(\Lambda)|$ holds.

Proof. Observe first that $|P(\Lambda)| \leqslant |P(\Xi)|$, since any solution has at most $|P(\Xi)|$ different values. Thus, it is necessary that $|P(\Lambda)| = |P(\Xi)|$ for all solutions to be q-periodic.

Conversely, if $|P(\Lambda)| = |P(\Xi)|$, then the solution of maximal diversity is already of period q, hence all solutions are periodic. □

We have seen that a periodic DI-sequence w.r.t. W corresponds to the orbit structure of the pseudo-inverse $\Lambda(s) = ps + w_0$ as a map from $\mathbb{Z}_q$ to $\mathbb{Z}_q$, where $q = \gcd(w_0 - W)$. We therefore study the group action of the group $\langle \Lambda \rangle$ generated by Λ more closely.

Definition 6. Let p be an integer such that $|p| \geqslant 2$ and let $q \in \mathbb{N}$ such that $\gcd(p, q) = 1$. Furthermore, let $\Lambda : \mathbb{Z}_q \to \mathbb{Z}_q$ be defined as $\Lambda(s) = ps + w_0$, $w_0 \in \mathbb{Z}$. A map $g : \mathbb{Z}_q \to \mathbb{Z}$ is called a *solution* for Λ if $g(\Lambda(s)) = g(s)$ holds for all $s \in \mathbb{Z}_q$.

If $g : \mathbb{Z}_q \to \mathbb{Z}$ is a solution for Λ, then g defines a periodic DI-sequence $f_g(X)$ by a periodic continuation to the left and right, i.e.,

$$f_g(X) = \sum_{j \in \mathbb{Z}} X^{jq}(g_0 + g_1 X + \cdots + g_{q-1} X^{q-1}).$$

Like for DI-sequences we are interested in solutions g for Λ such that $\{g(i) \mid i \in \mathbb{Z}_q\}$ is as large as possible.

Definition 7. Let $(p, q) = 1$ and let $\Lambda : \mathbb{Z}_q \to \mathbb{Z}_q$ be defined as $\Lambda(s) = ps + w_0$. A solution $g : \mathbb{Z}_q \to \mathbb{Z}$ (for Λ) is called *Λ-maximal* if $|g(\mathbb{Z}_q)|$ equals $|P(\Lambda)|$.

A Λ-maximal solution assigns to each periodic orbit of Λ a different value. The number $P(\Lambda)$ of different orbits of Λ is given by the Cauchy-Frobenius Lemma, cf., [15], as

$$|P(\Lambda)| = \frac{1}{\Delta} \sum_{j=0}^{\Delta - 1} |\mathrm{Fix}(\Lambda^j)|,$$

where Δ is the order of $\langle \Lambda \rangle$ and $\mathrm{Fix}(\Lambda^j)$ is the set of fixed points of Λ^j.

The number $|\mathrm{Fix}(\Lambda^j)|$ can be computed with the next lemma.

Lemma 5. *Let* $(r,q)=1$ *and define* $A:\mathbb{Z}_q\to\mathbb{Z}_q$ *as* $A(s)=rs+w_0$. *The number of fixed points of* A^j *is given by*

$$|\mathrm{Fix}(A^j)| = \begin{cases} \gcd(r^j-1,q) & \text{if } \gcd(r^j-1,q)\,|\,(1+\cdots+r^{j-1})w_0 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. One computes that $A^j(s)=r^js+(1+r+r^2+\cdots+r^{j-1})w_0 \bmod q$, so fixed points of A^j are solutions of the equation $(r^j-1)s\equiv -(1+r+r^2+\cdots+r^{j-1})w_0 \bmod q$. Now the assertion follows from Theorem 57 in [13]. □

In [7], the authors discuss several properties of periodic DI-sequences. The main open problem can be phrased like this: If the DI-equation allows a period q, i.e., $q=\gcd(W-w_0)$, does there always exist a solution of minimal period q? The next theorem provides an answer.

Theorem 5. *Let* D *be a divisor of* $\gcd(w_0-w_1,\ldots,w_0-w_{|p|-1})$, *and let* $\delta=\mathrm{ord}(p\in\mathbb{Z}_D^*)$, *where* $\mathbb{Z}_D^*$ *is the set of units in the ring* $\mathbb{Z}_D$. *Furthermore, let*

$$\begin{aligned} \Lambda &: \mathbb{Z}_D\to\mathbb{Z}_D,\\ s &\mapsto ps+w_0. \end{aligned}$$

(1) *If there exist an* $s\in\mathbb{Z}_D$ *such that* $\Lambda(s)=s$, *then a* Λ*-maximal solution* $g:\mathbb{Z}_D\to\mathbb{Z}$ *defines a periodic DI-sequence of minimal period* D.
(2) *If* $\Lambda^\delta(s)=s$ *for all* $s\in\mathbb{Z}_D$, *then a* Λ*-maximal solution* $g:\mathbb{Z}_D\to\mathbb{Z}$ *defines a DI-sequence of minimal period* D.
(3) *If* $\Lambda^\delta(s)=s+v$ *for all* $s\in\mathbb{Z}_D$, *where* $v\neq 0$, *then a* Λ*-maximal solution* $g:\mathbb{Z}_D\to\mathbb{Z}_q$ *induces a DI-sequence of period* $\gcd(v,D)<D$.

Proof. Let $\Lambda(s_0)=s_0$ for a $s_0\in\mathbb{Z}_D$. Since $\Lambda:\mathbb{Z}_D\to\mathbb{Z}_D$ is bijective, a Λ-maximal solution $g:\mathbb{Z}_D\to\mathbb{Z}$ takes the value $g(s_0)$ only at the point s_0. Therefore the periodic extension $f_g(X)$ of g has minimal period D.

Part 3 is obvious: If both v and D are periods of $f_g(X)$, then the actual period divides both v and D.

The hard part of the theorem is the second assertion. It is based on Lemma A.1 in the appendix. Let us assume that $g:\mathbb{Z}_D\to\mathbb{Z}$ is a Λ-maximal solution, i.e., $|g(\mathbb{Z}_D)|=|P(\Lambda)|$, such that the minimal period the sequence $f_g(X)$ induced by g is $\bar{q}$, a proper divisor of D, i.e., $D=\bar{q}r$. If π is a prime divisor of r, then the sequence $f(X)$ is also of period D/π. This implies that g satisfies $g(s+D/\pi)=g(s)=g(\tau(s))$, where $\tau(s)=s+D/\pi$, for all $s\in\mathbb{Z}_D$, i.e., g is constant on the orbits of the group action of the group $\langle\Lambda,\tau\rangle$ on $\mathbb{Z}_D$. It implies that $|g(\mathbb{Z}_D)|$ cannot exceed the number of orbits defined by the group action of $\langle \Lambda, \tau \rangle$ on $\mathbb{Z}_D$. The number of orbits is given by

$$\frac{1}{|\langle \Lambda, \tau \rangle|} \sum_{\lambda \in \langle \Lambda, \tau \rangle} |\mathrm{Fix}(\lambda)|,$$

due to the Cauchy–Frobenius Lemma, cf. [15],

Together with Lemma A.1 in the appendix, it follows that

$$|g(\mathbb{Z}_D)| \leqslant \frac{1}{|\langle \Lambda, \tau \rangle|} \sum_{\lambda \in \langle \Lambda, \tau \rangle} |\mathrm{Fix}(\lambda)| < |P(\Lambda)|,$$

a contradiction to the Λ-maximality of g. $\square$

Example. (1) Let $p = 19$ and let W be a 19-residue set such that $\gcd(w_0 - W) = 50$. The period of a periodic DI-equation is 50. Since $\Lambda(s) = 19s + w_0 \bmod 50$ and $19^{10} \equiv 1 \bmod 50$ as well as $\Lambda^{10}(s) = s \bmod 50$, we see that a periodic Λ-maximal DI-sequence has minimal period 50.

(2) Let $p = 3$, the 3-residue set $W = \{1, 17, 81\}$ has $q = \gcd(1 - W) = 16$. Therefore a possible period of a DI-sequence for W is 16. Thus we study $\Lambda(s) = 3s + 1 \bmod 16$. Since $3^4 \equiv 1 \bmod 16$ and since $\Lambda^4(s) = s + 8$, we conclude that the period of a Λ-maximal solution is $\gcd(8, 16) = 8$. Thus we study $\Lambda'(s) = 3s + 1 \bmod 8$. Since $3^2 \equiv 1 \bmod 8$ and $(\Lambda')^2(s) = s + 4 \bmod 8$, conclude that the period is 4, rather than 8. Finally, for $\Lambda''(s) = 3s + 1 \bmod 4$ we have $3^2 \equiv 1 \bmod 4$ and $(\Lambda'')^2(s) = s \bmod 4$ which shows that a maximal Λ-maximal solution extends to a periodic DI-sequence of minimal period 4.

5. DI-equations having only periodic solutions

As we have seen, one can, under mild conditions on the residue set W, construct DI-sequences which are periodic and which have minimal period greater than 1.

In this section, we tackle the problem of describing p-residue sets W which have the property that the solution of maximal diversity of $f(X) = Q_W(X) f(X^p)$ is already periodic.

If W is a complete digit set, cf. [14, 16, 17], then the solution of maximal diversity of $f(X) = Q_W(X) f(X^p)$ has period one, i.e., it is the constant sequence. Guided by this observation, we define what we like to call quasi-complete digit sets. A p-residue set W is called quasi-complete if the solution of maximal diversity of $f(X) = Q_W(X) f(X^p)$ is already periodic.

We show that quasi-complete digit sets exist for every $p \in \mathbb{Z}$, $|p| \geqslant 2$. Like for the construction of complete digit sets, it is very difficult to present a general method how to construct quasi-complete digit sets.

Definition 8. Let W be a p-residue set, W is called a *complete digit set* (for p) if and only if for any $l \in \mathbb{Z}$ there exist a $K \in \mathbb{N}$ and $w_j \in W$, $j = 0, \ldots, K$ such that $w_K \neq 0$

and

$$l = \sum_{j=0}^{K} w_j p^j.$$

The meaning of a complete digit set is clear: Any $l \in \mathbb{Z}$ has a unique finite p-adic expansion with digits in W. In particular, $0 \in W$ is necessary for being a complete digit set.

In terms of the image-part and remainder-part map, the jth digit of $l \in \mathbb{Z}$ is given by $w_j = \zeta(\Xi^j(l))$, which is nothing else than the Euclidean algorithm.

If Ξ has periodic points different from zero, then the periodic point has no representation as given in Definition 8.

Example. Let $p = 3$, then $W = \{-1, 0, 7\}$ is a complete digit set, while $W = \{-1, 0, 4\}$ is not a complete digit set since $\Xi(-2) = -2$.

By our above considerations we have the following characterization of complete digit sets.

Lemma 6. *Let W be a p-residue set such that $0 \in W$. The following statements are equivalent*:
(1) *W is a complete digit set.*
(2) *The image-part map (w.r.t. W) $\Xi : \mathbb{Z} \to \mathbb{Z}$ has 0 as the only periodic point.*
(3) *The DI-equation $f(X) = Q_W(X) f(X^P)$ has only period one solutions.*

Proof. We start by showing that (1) implies (2). Let $l \in \mathbb{Z}$ and $l = \sum_{j=0}^{K} w_j p^j$, then we have $\Xi(l) = \sum_{j=1}^{K} w_j p^{j-1}$ and therefore $\Xi^{K+1}(l) = 0$.

Implication (2) yields (3) as a consequence of Theorem 2. There is only one generating set, namely $S = \{0\}$.

Finally, we prove that (3) implies (1). Since $0 \in W$, we know that $0 \in \mathrm{Per}(\Xi)$. Since every solution $f(X)$ has period one, it follows that there exist no other periodic points of Ξ. $\square$

It is the third characterization of complete digit sets which leads to the following definition.

Definition 9. The p-residue set W is called *quasi-complete* digit set if all solutions of the DI-equation

$$f(X) = Q_W(X) f(X^p)$$

are periodic.

As a next step we establish the existence of certain quasi-complete digit sets.

Theorem 6. *Let* $p \geqslant 2$ *and let* $q \in \mathbb{N}$ *such that* $\gcd(p,q)=1$. *The* p-*residue set* $W=\{w_0+jq \mid j=0,\dots,p-1\}$ *with* $w_0 \in \mathbb{Z}$ *is a quasi-complete digit set, if and only if*

$$\frac{w_0}{p-1} \notin \mathbb{Z}.$$

Proof. By Corollary 1, the core interval $C(W)$ is contained in

$$\left[-\frac{w_0}{p-1}-q, -\frac{w_0}{p-1}\right].$$

Thus a solution of maximal diversity is defined by the different periodic orbits of Ξ on the set $U=[-(w_0+(p-1)q)/(p-1), -w_0/p-1] \cap \mathbb{Z}$.

If $-w_0/(p-1) \notin \mathbb{Z}$, then the set U contains q successive elements of $\mathbb{Z}$ and forms a residue set for $\mathbb{Z}_q$. By Corollary 2, it follows that $p\Xi(s)+w_0 \equiv s \bmod q$ for all $s \in U$. Therefore, it follows that the number of different periodic orbits of $\Xi : U \to U$ is equal to the number of different periodic orbits of $\Lambda : \mathbb{Z}_q \to \mathbb{Z}_q$. Applying Theorem 4, we conclude that all solutions of the associated DI-equation are periodic.

Now, let us assume that $-w_0/(p-1) \in \mathbb{Z}$ and that $f(X)$ is a solution of maximal diversity and of period $q=\gcd(w_0-W)$, i.e., all solutions are of period q. Since $-w_0/(p-1)-q$ and $-w_0/(p-1)$ are in $\mathbb{Z}$, they are fixed points of Ξ. It follows that the core interval $C(W)$ is contained in

$$\left[-\frac{w_0}{p-1}-q, -\frac{w_0}{p-1}\right].$$

That proves that a solution $f(X)$ of maximal diversity has different values at $-w_0/(p-1)$ and $-w_0/(p-1)-q$ and is therefore not of period q; contrary to our assumption. □

For negative values of p the situation is slightly different.

Theorem 7. *Let* $p \leqslant -2$ *and* $q \in \mathbb{N}$ *such that* $\gcd(p,q)=1$. *The* p-*residue set* $W=\{w_0+jq \mid j=0,\dots,|p|-1\}$ *with* $w_0 \in \mathbb{Z}$, *is a quasi-complete digit set.*

Proof. The set $W^{(2)}=\{w+pv \mid v,w \in W\}$ is a p^2-residue set and of the form $\{w^*+jq \mid j=0,\dots,p^2-1\}$ where $w^*=p(w_0+(|p|-1)q)+w_0$ (the smallest element in $W^{(2)}$). By Corollary 1, we conclude that

$$C(W) \subset \left[-\frac{w^*}{p^2-1}-q, -\frac{w^*}{p^2-1}\right].$$

We apply Theorem 6 for $p^2>0$ and $W^{(2)}$. If $w^*/(p^2-1) \notin \mathbb{Z}$, then every solution of the equation $f(X)=Q_{W^{(2)}}(X)f(X^{p^2})$ is periodic. Now, observe that $Q_{W^{(2)}}(X)=Q_W(X)Q_W(X^p)$ and that every solution of the equation $f(X)=Q_W(X)f(X^p)$ is also a solution of the equation $f(X)=Q_{W^{(2)}}(X)f(X^{p^2})$, which shows that $f(X)$ is periodic.

We now assume that $w^*/(p^2-1)\in\mathbb{Z}$, i.e., $-w^*/(p^2-1)$ and $-w^*/(p^2-1)-q$ are fixed points of Ξ^2, the second iterate of the image-part map (w.r.t. W). An elementary computation shows that $\Xi(-w^*/(p^2-1)) = -w^*/(p^2-1)-q$.

It follows that $C(W)$ is contained in an interval of length q such that the endpoints $a=-w^*/(p^2-1)-q$ and $b=-w^*/(p^2-1)$ form a Ξ-cycle of period two. We can now apply Theorem 4 to prove the periodicity of a DI-sequence with maximal diversity. Note that $b-a=q$, which implies that $a \bmod q$ is a fixed point of the pseudo-inverse Λ. Furthermore, we have that $\Xi(]a,b[\cap\mathbb{Z}) =]a,b[\cap\mathbb{Z}$ and $\Lambda(\Xi(s))\equiv s \bmod q$ for all $s\in]a,b[\cap\mathbb{Z}$. Therefore we have $|P(\Lambda)|=|P(\Xi)|$, i.e., any solution is periodic. $\square$

Theorems 6 and 7 establish the existence of quasi-complete digit sets for all $|p|\geqslant 2$. These digit sets are special, since they are part of an arithmetic progression. As we shall see later, there exist also quasi-complete digit sets W not part of an arithmetic progression. Like for complete digit sets, no general method is known for constructing quasi-complete digit sets.

Lemma 7. *Let $W=\{w_0,\dots,w_{|p|-1}\}$ be a p-residue set and let Ξ be the associated image-part map. If $|\mathrm{Per}(\Xi)| = \gcd(w_0-W)$, then W is a quasi-complete digit set.*

Proof. Let $q=\gcd(w_0-W)$. By Lemma 4, we have that $\mathrm{Per}(\Xi)$ contains a q-residue set. Since $|\mathrm{Per}(\Xi)|=q$, it follows that $\mathrm{Per}(\Xi)$ is a q-residue set. That means that the dynamics of Ξ on $\mathrm{Per}(\Xi)$ is completely described by the dynamics of the pseudo-inverse Λ on $\mathbb{Z}_q$. We conclude that we have $|P(\Lambda)|=|P(\Xi)|$. By Theorem 4, our assertion is proved. $\square$

Example. (1) The converse of Lemma 7 is not true, as the following example shows. Let $p=4$ and let $W=\{-9,-3,0,6\}$. Then we have that $\gcd(0-W)=3$, all solutions of $f(X)=Q_W(X)f(X^4)$ are periodic and $|\mathrm{Per}(\Xi)|=6>3$.

(2) The 3-residue set $W=\{1,8,9\}$ is a quasi-complete digit set since $\mathrm{Per}(\Xi)=\{-4\}$. Although every solution of the associated DI-equation has period one, W is not a complete digit set.

However, if W is a p-residue set such that $\mathrm{Per}(\Xi)=\{j_0\}$, then $W+(p-1)j_0$ is a complete digit set (see Lemma 3 and Theorem 6).

(3) Let $W_k=\{1,5,12k+9\}$, where $k\in\mathbb{Z}$. For any $k\in\mathbb{Z}$ the set W_k is a 3-residue set and we have $4=\gcd(1-W_k)$. A numerical computation (for $k=0$ up to $k=5460$) shows that for $k=0,1,9,38,3294$, the set W_k is a quasi-periodic digit set and not a complete digit set. Indeed, we have $|P(\Xi_k)|=2$ for the selected values of k.

On the other hand, for $k=1640,2214,2460$ we have $|P(\Xi_{W_k})|=31,37,56$, respectively. For the remaining values of k we observe that $|P(\Xi_{W_k})|$ is varying about 10.

It is not known whether there are infinitely many values of k such that $|P(\Xi_{W_k})|=2$.

(4) Let $W_k=\{-1,0,6k+1\}$, where $k\in\mathbb{Z}$. Then it well known, see [17], that for infinitely many values of k the set W_k is a complete digit set. E.g., whenever $6k+1$ is of the form 3^j-2, then W_k is a complete residue set.

(5) Any 2-residue set is (after a suitable shift) of the form $W_k = \{0, 2k+1\}$, where $k \in \mathbb{Z}$, i.e., W_k is part of an arithmetic progression. By Theorem 6, it follows that W_k is not a quasi-complete residue set.

(6) Any (-2)-residue set is part of an arithmetic progression and therefore, by Theorem 7, any (-2)-residue set is a quasi-complete digit set.

6. Right-periodic DI-sequences

As several examples indicate, there not only exist periodic DI-sequences but also DI-sequences which are periodic to the left or to the right without being periodic itself. In this section we will study under which conditions right- or left-periodic DI-sequences exist. Moreover, we will give a necessary and sufficient condition on W for the existence of a right- and left-periodic DI-sequence that is not periodic.

Since a right-periodic DI-sequence $f(X)$ induces the left periodic DI-sequence $f(X^{-1})$, it suffices to study right-periodic DI-sequences.

Definition 10. A DI-sequence $f(X)$ w.r.t. W is called *right-periodic* if there exists an $N \in \mathbb{Z}$ and a $d \in \mathbb{N} \setminus \{0\}$ such that

$$f_{n+d} = f_n$$

holds for all $n \geqslant N$.

Like for periodic sequences, we say that the DI-sequence has minimal period D if D is minimal among all the numbers $d \geqslant 1$ such that $f_{n+d} = f_n$ holds for all $n \geqslant N$.

Example. Let $p = 2$ and $W = \{0, 5\}$. The generating sets are given by $\{0, s, 5\}$ with $s \in \{1, 2, 3, 4\}$. If we define the values for the generating set $\{0, 1, 5\}$ as $f_0 = 0$, $f_1 = 1$ and $f_5 = 2$, then they extend to a DI-sequence that is left- and right-periodic and not periodic.

If $f(X)$ is a right-periodic sequence, one might ask for the minimal value of N such that $f_{n+d} = f_n$ holds for all $n \geqslant N$. For DI-sequences (w.r.t. W) there exists a universal N, depending only on the residue set W.

Lemma 8. *Let $W = \{w_0 < \cdots < w_{p-1}\}$ be a p-residue set and $p \geqslant 2$. If $f(X)$ is a right-periodic DI-sequence (w.r.t. W) of period D, then*

$$f_{n+D} = f_n$$

holds for all $n > -w_{p-1}/(p-1)$.

Proof. Let N be such that $N > -w_{p-1}/(p-1) + 1$ and $f_{n+D} = f_n$ holds for all $n \geqslant N$. Then, due to the choice of N, we have that

$$p(N-1) + w_{p-1} \geqslant N.$$

Since $f(X)$ is a DI-sequence it follows that $f_{p(N-1)+w_{p-1}} = f_{N-1}$. Now we consider f_{N-1+D}. Since $f(X)$ is a DI-sequence, we obtain $f_{N-1+D} = f_{p(N-1+D)+w_{p-1}}$ which is the same as $f_{p(N-1)+w_{p-1}}$, due to the right-periodicity. Thus $f_{N-1} = f_{N-1+D}$, and therefore $f_n = f_{n+D}$ for all $n \geqslant N-1$. An induction argument completes the proof. □

An inspection of the proofs of Theorem 3 and Corollary 2 shows that these proofs also work for the case of right-periodicity. This results in

Lemma 9. *If $f(X)$ is a right-periodic DI-sequence (w.r.t. W) of period D, then D is a divisor of* $\gcd(w_0 - W)$. *In particular,* $\gcd(p, D) = 1$.

Moreover, for all $j \in \mathbb{Z}$

$$p\Xi(j) + w_0 \equiv j \bmod D$$

holds.

The existence of a right-periodic sequence that is not a periodic sequence is guaranteed by the next theorem.

Theorem 8. *Let $p \geqslant 2$ be a natural number. There exists a right-periodic but non-periodic DI-sequence (w.r.t. W) $f(X)$ if and only if W satisfies*

(1) $|P(\Xi)| \geqslant 2$,

(2) $w_{p-1}/(p-1) \in \mathbb{Z}$.

Proof. We begin with the first part, namely, if a right-periodic and non-periodic solution exists, then (1) and (2) are true. Let $W = \{w_0 < \cdots < w_{p-1}\}$ be such that the DI-equation $f(X) = Q_W(X) f(X^p)$ has a right-periodic solution $f(X)$ of minimal period D.

If $|P(\Xi)| = 1$, then every solution is of period 1. We may therefore assume that $|P(\Xi)| \geqslant 2$.

Due to Lemma 9, we have that

$$p\Xi(s) + w_0 \equiv s \bmod D.$$

Let us further assume that $w_{p-1}/(p-1) \notin \mathbb{Z}$. By Lemma 8, we have that $f_n = f_{n+D}$ holds for all $n > -w_{p-1}/(p-1)$. The core interval $C(W)$ satisfies $C(W) \subset]-w_{p-1}/(p-1), \infty[$.

If $s_1,\ s_2 \in \mathbb{Z}$ such that $s_1 \equiv s_2 \bmod D$, then there exist M such that $\Xi^M(s_1)$ and $\Xi^M(s_2)$ are both in $C(W)$. By Corollary 2, it follows $\Xi^M(s_1) \equiv \Xi^M(s_2) \bmod D$. Therefore the sequence is periodic, which is a contradiction.

Finally we prove the assertion that (1) and (2) imply the existence of a right-periodic but non-periodic sequence. Now let us assume that $W = \{w_0 < \cdots < w_{p-1}\}$ satisfies the above two assumptions. Then the point $-w_{p-1}/(p-1)$ is a fixed point of Ξ, and the core interval satisfies

$$C(W) \subset \left[-\frac{w_{p-1}}{p-1}, -\frac{w_0}{p-1}\right].$$

Moreover, for $j\in\mathbb{Z}$ such that $j>-w_{p-1}/(p-1)$, we have that $\Xi^n(j)>-w_{p-1}/(p-1)$ for all $n\in\mathbb{N}$ and there exists an $N\in\mathbb{N}$ such that $\Xi^N(j)\in C(W)\setminus\{-w_{p-1}/(p-1)\}$. We define a right-periodic non-periodic DI-sequence by setting

$$f_j=\begin{cases}0 & \text{if } \Xi^N(j)=-\frac{w_{p-1}}{p-1} \text{ for some } N\in\mathbb{N},\\ 1 & \text{otherwise.}\end{cases}$$

Since $f_{-w_{p-1}/(p-1)}=0$ and $f_{-w_{p-1}/(p-1)+l}=1$ for all $l>0$, it follows that $f(X)$ is right-periodic but non-periodic. □

For the left-periodicity we have a similar result.

Corollary 3. *Let $W=\{w_0<\cdots<w_{p-1}\}$ be a p-residue set and $p\geqslant 2$. If $f(X)$ is a left-periodic sequence DI-sequence (w.r.t. W), then $f_n=f_{n-D}$ holds for all $n<w_0/(p-1)$.*

Moreover, a left-periodic but non-periodic DI-sequence (w.r.t. W) exists if and only if W satisfies
(1) $|P(\Xi)|\geqslant 2$,
(2) $w_0/(p-1)\in\mathbb{Z}$.

So far we have only considered values of p which are positive. The following lemma justifies why it is sufficient to study the case $p>0$.

Lemma 10. *If $p\leqslant -2$ and if $f(X)$ is a right-periodic DI-sequence w.r.t. the p-residue W set, then $f(X)$ is periodic.*

Proof. Let us assume that $f(X)$ is a right-periodic sequence of minimal period D and a solution of $f(X)=Q_W(X)f(X^p)$. Obviously, $f(X)$ is also a solution of the DI-equation $f(X)=Q_W(X)Q_W(X^p)f(X^{p^2})=Q_{W^{(2)}}(X)f(X^{p^2})$ and $f(X)$ is right-periodic. Therefore Lemma 9 implies $\gcd(D,p^2)=1$ which gives $\gcd(D,p)=1$. Lemma 9 gives that

$$W^{(2)}=\{w_0<w_1=w_0+D\alpha_1<\cdots<w_{p^2-1}=w_0+D\alpha_{p^2-1}\},$$

where $\alpha_{i+1}-\alpha_i\geqslant 1$ for all $0<i<p^2-1$. By Theorem 8, applied to $p^2>0$ and $W^{(2)}$, we conclude that $w_{p^2-1}/(p^2-1)\in\mathbb{Z}$. Due to the remark following Corollary 1, $-w_{p^2-1}/(p^2-1)$ is a fixed point of Ξ^2 and, moreover, it is the left point of the core interval $C(W)$, i.e.,

$$C(W)\subset[a,b],$$

where $a=-w_{p^2-1}/(p^2-1)$ and $b=-w_0/(p^2-1)$.

To prove that $b\in\mathbb{Z}$, or, equivalently that b is a fixed point of Ξ^2, we consider the sequence $\partial_0(f(X))$. Since $f(X)$ is a p-DI-sequence, we have that $\partial_0(f(X))=x^{\kappa_0}f(X)$ for a certain value of $\kappa_0\in\mathbb{Z}$. Since $f(X)$ is right-periodic and $p<0$ it follows that

$\partial_0(f(X))$ is left periodic. Therefore $f(X)$ is left- and right-periodic. Corollary 3 applied to $f(X)$ and $W^{(2)}$ then shows that b is indeed a fixed point of Ξ^2.

We now compute $w_{p^2-1} - w_0$ as

$$(w_0 + D\alpha_{p^2-1}) - w_0 = \sum_{i=0}^{p^2-2} (w_{i+1} - w_i) \geqslant D(p^2 - 1).$$

This shows that $b - a \geqslant D$. Firstly, we deal with the case $b - a > D$. We suppose that s_1 and s_2 are such that $s_1 \equiv s_2 \bmod D$ and $s_1 < b$ and $s_2 > a$. Then there exist $r_1, r_2 \in \mathbb{Z}$ such that $s_i + r_iD \in]a, b[$ for $i = 1, 2$. Due to the left-/right-periodicity we have

$$f_{s_1} = f_{s_1+r_1D},$$
$$f_{s_2} = f_{s_2+r_2D}$$

and since $s_i + r_iD \in]a, b[$ for $i = 1, 2$ we have $f_{s_1+r_1D} = f_{s_2+r_2D}$. Therefore $f(X)$ is periodic.

The case $b - a = D$ is studied in a similar way. We either have $s_1 + r_1D = s_2 + r_2D \in]a, b[$, which implies $f_{s_1} = f_{s_2}$; or we have $s_1 + r_1D = a$ and $s_2 + r_2D = b$ which again implies $f_{s_1} = f_{s_2}$ due to the fact that $\Xi(a) = b$ and $f(X)$ is a DI-sequence.

Therefore any right-periodic DI-sequence for negative p is a periodic sequence. □

The proof of Lemma 10 raises the question whether there exist DI-sequences $f(X)$ which are right- and left-periodic, but not periodic. As a consequence of the above proof, we have

Theorem 9. *Let W be a p-residue set and $p \geqslant 2$. A left- and right-periodic but non-periodic DI-sequence w.r.t. W exists if and only if W is of the form $W = \{w_0 + jq \mid j = 0, \ldots, p-1\}$ and $w_0/(p-1) \in \mathbb{Z}$.*

Proof. If $f(X)$ is left/right-periodic of period D, then

$$C(W) \subset [a, b],$$

where $a = -w_{p-1}/(p-1)$ and $b = -w_0/(p-1)$ and a and b are fixed points of Ξ, as a consequence of Theorem 8 and Corollary 3. If we replace p^2 by $p > 0$ and $W^{(2)}$ by W, then the same arguments as in the proof of Lemma 10 yield that $b - a \geqslant D$ and also that $b - a > D$ implies the periodicity of $f(X)$. Thus $b - a = D$. Since W allows a periodic solution we have

$$W = \{w_0 < w_1 = w_0 + D\alpha_1 < \cdots < w_{p-1} = w_0 + D\alpha_{p-1}\},$$

and since $(1-p)(b-a) = w_0 - w_{p-1}$ we have

$$w_{p-1} - w_0 = (p-1)D = \sum_{i=0}^{p-2} (w_{i+1} - w_i) = D \sum_{i=0}^{p-2} (\alpha_{i+1} - \alpha_i)$$

which implies $w_{i+1} - w_i = D$ for all $0 \leqslant i < p-1$. Thus W is of the desired form.

On the other hand, if W is of the given form, then a and b as given in the beginning of the proof are elements of a generating set S. A right- and left-periodic but non-periodic solution $f(X)$ is defined by setting $f_a=1$, $f_b=2$ and $f_s=0$ for all remaining elements of S. $\square$

7. Reflection symmetric solutions

As introduced in [6] we study DI-sequences which show an additional symmetry, which can be expressed as $f(X)=X^a f(X^{-1})$, where $a\in\mathbb{Z}$, i.e., the sequence is invariant under reflection about 0 followed by a shift. In [6], it was conjectured that a DI-sequence with this type of symmetry is already periodic.

In this section we discuss the conjecture in full detail. In particular, we show that for the symmetry $f(X)=X^a f(X^{-1})$ and a an even number, the conjecture is true for all $|p|\leqslant 3$ and for all $p\equiv 0 \bmod 2$. The conjecture is false for all odd p such that $|p|\geqslant 4$.

For the symmetry $f(X)=X^a f(X^{-1})$ with a odd, we show that the conjecture is true for all $|p|\leqslant 4$ and false for all $|p|\geqslant 5$.

We begin with the remark that it suffices to study D-sequences with symmetry $f(X)=X^{\varepsilon} f(X^{-1})$, where $\varepsilon=0,1$. Suppose $f(X)=X^a f(X^{-1})$, then for $f(X)=X^b g(X)$ we obtain

$$X^b g(X) = X^a X^{-b} g(X^{-1}).$$

Thus, for a even, $g(X)$ satisfies $g(X)=g(X^{-1})$ (symmetry of type 1), and for a odd, $g(X)$ satisfies $g(X)=Xg(X^{-1})$ (symmetry of type 2). By Lemma 3, the sequence $g(X)$ is a DI-sequence. The following lemma leads to a necessary criterion for the existence of periodic DI-sequence of symmetry type 1.

Lemma 11. *Let p be an integer such that $|p|\geqslant 2$ and W a p-residue set. Furthermore, let Ξ and ζ be the image-part and remainder-map w.r.t. W, respectively. If there exists a $v\in W$ such that $\Xi(-v)\neq 0$, then every solution of*

$$f(X) = Q_W(X) f(X^p)$$

which satisfies $f(X)=f(X^{-1})$ is periodic.

Proof. Since $f(X)=Q_W(X)f(X^p)$, we have $\partial_w(f(X))=f(X)$ for all $w\in W$. Since $f(X)=f(X^{-1})$, we have

$$\partial_w(f(X)) = \partial_w(f(X^{-1}))$$

for all $w\in W$. It remains to compute $\partial_w(f(X^{-1}))$ in terms of $\partial_w(f(X))$. To this end, we write

$$f(X) = \sum_{w\in W} X^w \partial_w(f)(X^p),$$

which yields

$$f(X^{-1}) = \sum_{w\in W} X^{-w}\partial_w(f)(X^{-p}) = \sum_{w\in W} X^{\zeta(-w)}X^{p\Xi(-w)}\partial_w(f)(X^{-p}),$$

where we used $-w=\zeta(-w)+p\Xi(-w)$. For $v\in W$ we obtain

$$\partial_{\zeta(-v)}(f(X^{-1})) = X^{\Xi(-v)}\partial_v(f)(X^{-1}).$$

Thus we conclude that

$$f(X) = \partial_{\zeta(-v)}(f(X)) = \partial_{\zeta(-v)}(f(X^{-1})) = X^{\Xi(-v)}\partial_v(f)(X^{-1}) = X^{\Xi(-v)}f(X),$$

i.e., $f(X)$ is of period $\Xi(-v)$. □

As a consequence of the above lemma we can state a necessary condition for the existence of a non-periodic DI-sequence with symmetry of type 1.

Corollary 4. *If W is a p-residue set such that the DI-equation $f(X)=Q_W(X)$ has a non periodic solution with type 1 symmetry, then $W=-W$.*

Proof. By Lemma 11, we must have $\Xi(-w)=0$ for all $w\in W$. Using $-w=\zeta(-w)+p\Xi(-w)$, we obtain $-w=\zeta(-w)$. In other words, the residue set has to be invariant under the map $x\mapsto -x$, for short $W=-W$. □

The symmetry of the residue set in Corollary 4 is reflected by $\Xi(-x)=-\Xi(x)$, i.e., Ξ is an odd function. This yields that $l\in\mathrm{Per}(\Xi)$ implies $-l\in\mathrm{Per}(\Xi)$, and the orbits generated by l and $-l$ are either the same or they are disjoint.

That observation allows us to construct generating sets S such that if $s\in S$ then either $-s\in S$ or $s\sim_{\Xi} -s$. We call such a generating set *adapted.* Any assignments of values to an adapted generating set S such that $f_s=f_{-s}$ whenever s and $-s$ belong to S, defines a DI-sequence of symmetry type 1.

Theorem 10. *If p is an even integer, $p\neq 0$, then every DI-sequence of symmetry type 1 is periodic.*

Proof. For a p-residue set W the condition $W=-W$ is necessary for the existence of a non-periodic DI-sequence with symmetry of type 1. Since $|W|$ is even and W is a p-residue set, no such W exists. Thus every DI-sequence with symmetry of type 1 is periodic. □

For p an odd number the construction of residue sets W that are invariant under the map $x\mapsto -x$ is of course possible.

Theorem 11. *If $p\in\{-3,3\}$, then every DI-sequence of symmetry type 1 is periodic.*

Proof. The residue set W has to be invariant under $x \mapsto -x$, thus $W = \{-k, 0, k\}$ where $k \in \mathbb{Z}$ and $k \not\equiv 0 \bmod 3$. If $p = -3$, then Theorem 7 applies, i.e., every DI-sequence (w.r.t. W) is periodic.

If $p = 3$ and $k/2 \notin \mathbb{Z}$, then, by Theorem 6, any DI-sequence (w.r.t. W) is periodic.

If $p = 3$ and $k/2 \in \mathbb{Z}$, then, by Theorem 9, the solution of maximal diversity is left- and right-periodic. Since the symmetry of type 1 requires $f_{-k/2} = f_{k/2}$ it follows that a DI-sequence with symmetry type 1 is periodic. □

Corollary 5. *For every odd p such that $|p| \geqslant 5$ there exists a non periodic DI-sequence of symmetry type* 1.

Proof. It suffices to construct one example. Let W be a p-residue set such that $-W = W$. Note that $0 \in W$. Moreover we assume that W contains $\{-(p^2-1), -2, 0, 2, p^2-1\}$. Then we have $\gcd(W) \leqslant 2$ and

$$\{-(p+1), 0, p+1\} \subset \mathrm{Fix}(\Xi).$$

Furthermore, there exist two periodic orbits, namely, $\Xi(1) = p$, $\Xi(p) = 1$ and $\Xi(-1) = -p$, $\Xi(-p) = -1$. Thus, an adapted generating set S contains the points $\{-(p+1), -1, 0, 1, p+1\}$. If we define $f(X)$ by setting $f_0 = 0$, $f_{-(p+1)} = f_{p+1} = 1$ and $f_1 = f_{-1} = 2$ and $f_s = 3$ for all other elements of S we obtain a DI-sequence $f(X)$ of symmetry type 1. It remains to show that $f(X)$ is not periodic. Since $\gcd(W) \in \{1, 2\}$, the only possible periods are 1 and 2. Obviously, $f(X)$ is not of period 1.

If p is an even number, then we have that $f_0 = 0$ and $f_p = f_1 = 2$, therefore $f(X)$ is not of period 2, thus not periodic at all.

If p is an odd number, then we have $f_0 = 0$ and $f_{(p+1)} = 1$, which shows that $f(X)$ is not of period 2, thus $f(X)$ is not at all periodic. □

Like for the symmetry of type 1 we start the study of DI-sequences of symmetry type 2 by giving a necessary condition for the existence of a periodic DI-sequence with symmetry type 2.

Lemma 12. *Let p an integer such that $|p| \geqslant 2$ and W a p-residue set. Furthermore, let Ξ and ζ be the image-part and remainder-map w.r.t. W, respectively.*

If there exist an $v \in W$ such that

$$\Xi(1 - v) \neq 1,$$

then every solution of

$$f(X) = Q_W(X) f(X^p)$$

which satisfies $f(X) = X f(X^{-1})$ is periodic.

Proof. Let $f(X)$ be a DI-sequence of symmetry type 2. Writing

$$f(X) = \sum_{w \in W} X^w \partial_w(f)(X^p),$$

we obtain

$$Xf(X^{-1}) = \sum_{w \in W} X^{-w+1} \partial_w(f)(X^{-p}) = \sum_{w \in W} X^{\zeta(-w+1)} X^{p\Xi(-w+1)} \partial_w(f)(X^{-p}).$$

In particular,

$$\partial_{\zeta(-v+1)}(Xf(X^{-1})) = X^{\Xi(-v+1)} \partial_v(f)(X^{-1}).$$

Since $f(X)$ is a DI-sequence (w.r.t. W) and of symmetry type 2, we conclude that

$$f(X) = X^{\Xi(-v+1)-1} f(X),$$

thus $f(X)$ is of period $\Xi(-v+1) - 1 \neq 0$. $\square$

Like for the existence of non-periodic solutions of symmetry type 1, the existence of non-periodic solution of type 2 requires an invariance property of the residue set.

Corollary 6. *If W is a p-residue set such that the equation $f(X) = Q_W(X) f(X^p)$ has a non-periodic solution with symmetry type* 2, *then*

$$W = 1 - p - W.$$

Proof. If $f(X)$ is a non-periodic DI-sequence of symmetry type 2, then $\Xi(1-w) = 1$ holds for all $w \in W$. This yields

$$1 - w = \zeta(1-w) + p\Xi(1-w) = \zeta(1-w) + p$$

for all w. Since $w = 1 - p - \zeta(1-w)$ and $w, \zeta(1-w) \in W$, it follows that W has to be invariant under the map $x \mapsto 1 - p - x$, for short $W = 1 - p - W$. $\square$

As a first result we can discuss the cases $|p| \leqslant 3$.

Theorem 12. *If $p \in \{-3, -2, 2, 3\}$, then every DI-sequence with symmetry type* 2 *is periodic.*

The proof follows the same lines as the proof of Theorem 11. By Corollary 12, a non-periodic solution can only exist if W is invariant under $x \mapsto 1 - p - x$. This forces W to be part of an arithmetic progression, which together with the symmetry of type 2 implies periodicity of a DI-sequence.

The invariance of W under the map $x \mapsto 1-p-x$ yields a symmetry for the associated image-part map Ξ.

Lemma 13. *Let W be invariant under the map $x \mapsto 1-p-x$, then the image-part map Ξ satisfies*

$$\Xi(1-x) = 1 - \Xi(x)$$

for all $x \in \mathbb{Z}$.

Proof. We have $x = \zeta(x) + p\Xi(x)$ which gives $1-p-x = -\zeta(x) + 1 - p - p\Xi(x)$. Now, we have that $\zeta(1-w) = -w+1-p$ holds for all $w \in W$, in particular, we have $-\zeta(x) + 1 - p = \zeta(1-\zeta(x))$. Therefore, we obtain

$$1 - x - p = \zeta(1-\zeta(x)) - p\Xi(x)$$

and thus

$$1 - x = \zeta(1-\zeta(x)) + p(1-\Xi(x))$$

which establishes the desired equality $\Xi(1-x) = 1 - \Xi(x)$. □

We have seen that type 1 symmetry induces a symmetry on $\mathrm{Per}(\Xi)$, namely $\mathrm{Per}(\Xi) = -\mathrm{Per}(\Xi)$. By Lemma 13, we conclude that a symmetry of type 2 yields $\mathrm{Per}(\Xi) = 1 - \mathrm{Per}(\Xi)$. This shows that $l \in \mathrm{Per}(\Xi)$ and $1 - l \in \mathrm{Per}(\Xi)$ are either Ξ-equivalent, i.e., in the same orbit, or that they define different orbits. Thus, we can construct generating sets S such that $s \in S$ implies either $(-s+1) \sim_{\Xi} s$ or $-s+1 \in S$. A generating set S having this property is called *adapted*. Any assignment of values to elements of an adapted set S such that $f_s = f_{-s+1}$ holds whenever s and $-s+1$ are elements in S, defines a DI-sequence of symmetry type 2.

Finally, we establish the existence of non-periodic DI-sequences of symmetry type 2. We begin with the construction of p-residue sets which satisfy $W = 1 - p - W$.

Lemma 14. *For every $p \in \mathbb{Z}$ such that $|p| \geqslant 4$ there exist p-residue sets W such that $W = 1 - p - W$.*

Proof. Let p be an even number and let $W^* \subset \mathbb{Z}$ of cardinality $|p|/2$ such that whenever $v, w \in W^*$ and $v \neq w$ it follows that $v - w \not\equiv 0 \bmod p$ and $v + w \not\equiv 1 \bmod p$. If $W^* \subset \mathbb{Z}$ is given in this way, then the set $W = W^* \cup (1 - p - W^*)$ is a p-residue set. Indeed, due to $v - w \not\equiv 0 \bmod p$, the set W^* is 'half' a p-residue set and we also have that $(1-w)-(1-v) \not\equiv 0 \bmod p$. The second condition ensures that $w-(1-v) \not\equiv 0 \bmod p$ for all $w, v \in W^*$. Note that $1 - p - (1 - p - x) = x$ for all $x \in \mathbb{Z}$, therefore it follows that W satisfies $W = 1 - p - W$.

If p is an odd number, then we start with $W^* = \{(1-p)/2\} \cup W'$ such that $|W'| = (|p|-1)/2$ and W^* satisfies the same two conditions as stated for the case that p is even. Then the set $W = W^* \cup (1 - p - W^*)$ is a p-residue set and satisfies $W = 1 - p - W$. □

Note that the construction in Lemma 14 is not possible for $|p| = 2$. It is possible for $|p| = 3$; but then it produces a residue set which is part of an arithmetic progression.

Corollary 7. *For any $p \in \mathbb{Z}$, $|p| \geqslant 4$ there exists a DI-sequence having a symmetry of type 2 and is non-periodic.*

Proof. Let $V = \{0, -(1-p)^2\}$ and consider a W^* as in the proof of Lemma 14 such that $V \subset W^*$. The set $W = W^* \cup (1 - p - W^*)$ is a p-residue set such that $W = 1 - p - W$. Moreover, W contains the points

$$\{0, 1-p, (1-p)(2-p), (1-p)(p-1)\},$$

which shows that $\gcd(0 - W)$ is a divisor of $1 - p$. Additionally, the image-part map Ξ has the fixed points 0, 1, $2 - p$, and $p - 1$. Thus, an adapted generating set S contains $\{0, 1, 2-p, p-1\}$. In order to have a symmetric solution, i.e., $f_n = f_{-n+1}$ we define $f_0 = f_1 = 1$ and $f_{2-p} = f_{p-1} = 2$ and $f_s = 3$ for all remaining elements of S. The sequence obtained by this assignment of values is a DI-sequence of symmetry type 2 and is not periodic. Indeed, a possible period is a divisor of $1 - p$. On the other hand, we have $f_{2-p} = 2$ and $f_{2-p+(1-p)} = f_1 = 1$ which shows that $f(X)$ is not periodic. $\square$

Example. For $p = 4$ the set W as given in the above proof is equal to $W = \{-9, -3, 6, 0\}$ which is indeed a 4-residue set. There exist three fixed points of Ξ, namely $-2, 0$, and 3 and one orbit of period two, namely $\{-1, 2\}$. Thus a non periodic DI-sequence of symmetry type 2 is defined by $f_{-2} = f_3 = 0$, $f_{-1} = 1$, $f_0 = f_1 = 2$.

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Appendix A.

In this section we state and prove the technical lemma used in the proof of Theorem 5.

Lemma A.1. *Let p and q be natural numbers such that $\gcd(p, q) = 1$. Furthermore, let $\delta = \operatorname{ord}(p \in \mathbb{Z}_q^*)$, i.e., the order of p in the multiplicative group of the units in the ring $\mathbb{Z}_q$. Let $w_0 \in W$ be such that the map*

$$\begin{aligned} &\Lambda : \mathbb{Z}_q \to \mathbb{Z}_q, \\ &s \mapsto ps + w_0 \end{aligned}$$

satisfies $\Lambda^\delta(x) = x$. *If* π *is a prime divisor of* q *and if* $\tau(s) = s + q/\pi$ *then we have*

$$|P(\Lambda)| > \frac{1}{|\langle \Lambda, \tau \rangle|} \sum_{\lambda \in \langle \Lambda, \tau \rangle} |\mathrm{Fix}(\lambda)|,$$

where $\langle \Lambda, \tau \rangle$ *denotes the group generated by* Λ *and* τ.

Proof. Since π is a prime divisor of q, we write $q = \pi^a q'$ and $\gcd(q', \pi) = 1$.

By the Cauchy–Frobenius Lemma, we have that

$$|P(\Lambda)| = \frac{1}{\delta} \sum_{j=0}^{\delta-1} |\mathrm{Fix}(\Lambda^j)|.$$

Since $\Lambda\tau = \tau^p \Lambda$ and $\gcd(p, \pi) = 1$, any $\lambda \in \langle \Lambda, \tau \rangle$ has a unique representation $\lambda = \tau^i \Lambda^j$, where $i = 0, \ldots, \pi - 1$ and $j = 0, \ldots, \delta - 1$. Therefore, the order of the group $|\langle \Lambda, \tau \rangle|$ is equal to $\pi\delta$. This gives

$$\frac{1}{|\langle \Lambda, \tau \rangle|} \sum_{\lambda \in \langle \Lambda, \tau \rangle} |\mathrm{Fix}(\lambda)| = \frac{1}{\delta\pi} \sum_{j=0}^{\delta-1} \sum_{i=0}^{\pi-1} |\mathrm{Fix}(\tau^i \Lambda^j)|$$

and we have to show that

$$\frac{1}{\delta\pi} \sum_{j=0}^{\delta-1} \sum_{i=0}^{\pi-1} |\mathrm{Fix}(\tau^i \Lambda^j)| < \frac{1}{\delta} \sum_{j=0}^{\delta-1} |\mathrm{Fix}(\Lambda^j)|. \tag{8}$$

We therefore need estimates of the number of fixed points of affine maps from $\mathbb{Z}_q$ onto $\mathbb{Z}_q$ which are of the form $A(s) = p^\alpha s + \beta$, $\alpha \in \mathbb{N}$ and $\beta \in \mathbb{Z}_q$. To this end we introduce a classification of these maps according to their possible number of fixed points. The map $A(s) = p^\alpha s + \beta$ is of *type* **1** if $\gcd(p^\alpha - 1, q)$ is not divisible by π^a, and we denote it by $T(A) = \mathbf{1}$. A is called of type **2**, for short $T(A) = \mathbf{2}$, if $\gcd(p^\alpha - 1, q)$ is divisible by π^a.

Now let $T(\Lambda^j) = \mathbf{1}$, then $T(\tau^i \Lambda^j) = \mathbf{1}$ for all $i \in \{0, \ldots, \pi - 1\}$ and we have

$$|\mathrm{Fix}(\Lambda^j)| = \frac{1}{\pi} \sum_{i=0}^{\pi-1} |\mathrm{Fix}(\tau^i \Lambda^j)|,$$

which is a consequence of Lemma 5.

If $T(\Lambda^j) = \mathbf{2}$, then $T(\tau^i \Lambda^j) = \mathbf{2}$ for all $i \in \{0, \ldots, \pi - 1\}$ and Lemma 5 yields that either

$$|\mathrm{Fix}(\tau^i \Lambda^j)| = 0$$

for all $i = 0, \ldots, \pi - 1$, or there exists a unique $i_0 \in \{0, \ldots, \pi - 1\}$ such that

$$|\mathrm{Fix}(\tau^i \Lambda^j)| = \begin{cases} \gcd(p^j - 1, q) & \text{if } i = i_0. \\ 0 & \text{otherwise.} \end{cases}$$

The left-hand side of Eq. (8) can be split into

$$\frac{1}{\delta\pi} \sum_{T(\Lambda^j) = \mathbf{1}} \sum_{i=0}^{\pi-1} |\mathrm{Fix}(\tau^i \Lambda^j)| + \frac{1}{\delta\pi} \sum_{T(\Lambda^j) = \mathbf{2}} \sum_{i=0}^{\pi-1} |\mathrm{Fix}(\tau^i \Lambda^j)|,$$

where the first sum is the sum over all maps in $\langle \Lambda \rangle$ which are of type **1** and the second sum is over all maps in $\langle \Lambda \rangle$ of type **2**.

If the first sum is over the empty set, then the sum is assumed to be zero. While the first sum may be zero, the second sum is always different from zero, since the identity map, $s \mapsto s$, is always of type **2**.

The sum over type **1** elements becomes

$$\frac{1}{\delta\pi} \sum_{T(\Lambda^j)=\mathbf{1}} \sum_{i=0}^{\pi-1} |\mathrm{Fix}(\tau^i \Lambda^j)| = \frac{1}{\delta} \sum_{T(\Lambda^j)=\mathbf{1}} |\mathrm{Fix}(\Lambda^j)|,$$

which follows from our above observation on type **1** maps.

It remains to show that

$$\frac{1}{\delta\pi} \sum_{T(\Lambda^j)=\mathbf{2}} \sum_{i=0}^{\pi-1} |\mathrm{Fix}(\tau^i \Lambda^j)| < \frac{1}{\delta} \sum_{T(\Lambda^j)=\mathbf{2}} |\mathrm{Fix}(\Lambda^j)|.$$

We begin with the following observation. There exists a $\Gamma \in \langle \Lambda \rangle$ such that

$$\langle \Gamma \rangle = \{\Lambda^j \mid T(\Lambda^j) = \mathbf{2},\ j \in \mathbb{N}\}.$$

The proof follows from the fact that there exists a minimal j_0 with $1 \leqslant j_0 \leqslant \delta$ such that π^a divides $\gcd(p^{j_0} - 1, q)$. This implies that π^a is a divisor of $\gcd(p^{j_0 k} - 1, q)$ for all $k \in \mathbb{N}$.

If we set $\Gamma = \Lambda^{j_0}$ we have to show

$$\frac{1}{\delta\pi} \sum_{j=0}^{\delta'-1} \sum_{i=0}^{\pi-1} |\mathrm{Fix}(\tau^i \Gamma^j)| < \frac{1}{\delta} \sum_{j=0}^{\delta'-1} |\mathrm{Fix}(\Gamma^j)|,$$

where δ' is the order of Γ. Note further that $\Gamma\tau = \tau\Gamma$, which follows from the fact that $\Gamma\tau = \tau^{p^{j_0}}\Gamma$ and $p^{j_0} \equiv 1 \bmod \pi^a$.

Let $p_j = \sum_{i=0}^{\pi-1} |\mathrm{Fix}(\tau^i \Gamma^j)|$ and let $\mathscr{F} = \{j \mid p_j \neq 0\}$. Furthermore, we define the subset $\mathscr{F}^*$ of $\mathscr{F}$ by setting $\mathscr{F}^* = \{j \mid p_j = |\mathrm{Fix}(\Gamma^j)| \neq 0\}$.

With these notations it remains to show

$$\frac{1}{\delta\pi} \sum_{j\in\mathscr{F}} p_j < \frac{1}{\delta} \sum_{j=0}^{\delta'-1} |\mathrm{Fix}(\Gamma^j)|.$$

If $\mathscr{F} = \mathscr{F}^*$, then we have

$$\frac{1}{\delta\pi} \sum_{j\in\mathscr{F}} p_j = \frac{1}{\delta\pi} \sum_{j=0}^{\delta'-1} |\mathrm{Fix}(\Gamma^j)| < \frac{1}{\delta} \sum_{j=0}^{\delta'-1} |\mathrm{Fix}(\Gamma^j)|,$$

the desired inequality.

From now on we consider the case $\mathscr{F} \neq \mathscr{F}^*$. Let j be in $\mathscr{F} \setminus \mathscr{F}^*$, i.e, there exists an $i \neq 0$ such that $\tau^i \Gamma^j$ has a fixed point. Then the map $(\tau^i \Gamma^j)^\pi = \Gamma^{j\pi}$ has also fixed points, i.e., $j\pi \bmod \delta'$ is in $\mathscr{F}^*$. That gives the inequality

$$p_j \leqslant p_{j\pi \bmod \delta'}$$

for all $j \in \mathscr{F}$.

Let us assume that π does not divide δ'. If $j \in \mathscr{F} \setminus \mathscr{F}^*$, i.e., $\tau^i \Gamma^j$, $i \neq 0$, has a fixed point, then the map $(\tau^i \Gamma^j)^\pi = \Gamma^{j\pi}$ has a fixed point. Since π does not divide δ', it follows that $\gcd(j\pi, \delta') = \gcd(j, \delta')$. Since $\Gamma^{j\pi}$ has a fixed point, it follows that $\Gamma^{\gcd(j,\delta')}$ has a fixed point, too. Since there exists a $k \in \mathbb{N}$ such that $(\Gamma^{\gcd(j,\delta')})^k = \Gamma^j$ it follows that $j \in \mathscr{F}^*$; a contradiction. I.e., if π does not divide δ' we always have $\mathscr{F} = \mathscr{F}^*$.

The final part of the proof is concerned with the case π divides δ'. Due to our assumption $\mathscr{F} \neq \mathscr{F}^*$, there exists at least one $j_0 \in \mathscr{F}$ such that $j_0 \neq 0$ and $j_0 \pi \equiv 0 \bmod \delta'$. For this particular j_0 we have the strict inequality $p_{j_0} < p_0$. Therefore,

$$\frac{1}{\delta\pi} \sum_{j \in \mathscr{F}} p_j < \frac{1}{\delta\pi} \sum_{j \in \mathscr{F}} p_{j\pi \bmod \delta'}.$$

If $j\pi \in \mathscr{F}^*$ for $j \in \mathscr{F}$, then there are at most π different values $j' \in \mathscr{F}$ such that $j'\pi \equiv j\pi \bmod \delta'$ holds. This gives the following estimates:

$$\begin{aligned} \frac{1}{\delta\pi} \sum_{j \in \mathscr{F}} p_j &< \frac{1}{\delta\pi} \sum_{j \in \mathscr{F}} p_{j\pi \bmod \delta'} \leqslant \frac{1}{\delta\pi} \sum_{j \in \mathscr{F}^*} \pi p_j \\ &= \frac{1}{\delta} \sum_{j \in \mathscr{F}^*} |\mathrm{Fix}(\Gamma^j)| \leqslant \frac{1}{\delta} \sum_{j=0}^{\delta'-1} |\mathrm{Fix}(\Gamma^j)|. \end{aligned}$$

This completes the proof. □

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